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# UNDERCONSTRAINED STRUCTURAL SYSTEMS

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**Abstract**—Underconstrained structural systems have fewer independent constraints than necessary to be geometrically invariant. Yet these systems can be statically indeterminate, with far-reaching implications for their kinematic mobility. In particular, such a system may possess a unique geometric configuration, in which it lacks finite mobility and allows a stable self-stress. The possibility of stable self-stress is shown to be a statical (as opposed to geometric) criterion for underconstrained systems with only infinitesimal kinematic mobility. Both local and global properties of underconstrained systems are investigated within the context of statical–kinematic interrelations. Among these, the relation between the equilibrium loads and configurations is of special interest as it underlies the concept of statically controlled geometry.

## INTRODUCTION

An analytical criterion of a geometrically invariant (stiff) system was formulated by Mobius[1] for an assembly of $n$ solids constrained in their motion by six external supports and interacting at $p$ points where mutual normal contact forces develop. Such an assembly is stiff when $p \geqslant 6(n-1)$. Stating immediately that this condition is necessary but not sufficient, Mobius proceeds with a thorough investigation of exceptional systems which satisfy his criterion but are not stiff. He reveals three interrelated properties of such systems: (a) they possess only infinitesimal mobility; (b) when the equations of equilibrium allow a solution, it is not unique; (c) each structural component has a maximum or minimum size compatible with other members of the assembly.

When formulating a stiffness criterion for pin-bar systems, Maxwell[2] reversed the roles of joints and bars by considering the former as nodal points and the latter as constraints. More interestingly, he recognized the existence of exceptional systems of the opposite nature: having fewer constraints than necessary, these systems are underconstrained and yet kinematically immobile. Maxwell associated such exceptional cases with the presence of maximum or minimum length bars.

Mohr[3] found that a maximum or minimum length is a general condition for the statical possibility of self-stress in kinematic chains, including underconstrained ones. Kinematic properties of underconstrained systems were studied in more detail by Levi-Civita and Amaldi[4]. In their monograph a structural system was modelled as an assembly of material points linked by ideal positional constraints representing the structural members. Then the kinematic properties of the system are fully determined by a compatible set of constraint equations

$$F_i(X_1,\dots,X_n,\dots,X_N) = 0, \quad i = 1,2,\dots,I. \tag{1}$$

The functions $F_i$ relate the $N$ generalized coordinates $X_n$ to the implicitly present geometric parameters of the system (member sizes, linear and angular distances, etc.). The linearized equations derived by differentiating eqn (1) at the solution point $X_n = X_n^0$ involve infinitesimal virtual displacements $x_n$

$$F_{in}^0 x_n = 0 \quad (F_{in} \equiv \partial F_i/\partial X_n). \tag{2}$$

Here and below a repeated (dummy) subscript denotes summation. By employing the principle of virtual work, equilibrium equations in the unknown generalized constraint reactions, $\Lambda_i$, are obtained from eqns (2)

$$F^0_{in}\Lambda_i = P_n. \tag{3}$$

When the rank $r$ of the Jacobian $F^0_{in}$ equals $N$, it follows from eqns (2) that all $x_n = 0$ and the system is geometrically invariant. For an underconstrained system, $r < N$ and the outcome depends on the relation between $r$ and $I$. When $r = I$ a non-zero solution for $x_n$ exists and the system allows displacements. At $r < I$, the equilibrium eqns (3), in the absence of external loads $P_n$, admit at least one set of $\Lambda_i \neq 0$. This set is used to form a linear combination of constraint equations expanded in power series

$$\Lambda_i[F^0_{in}x_n + (1/2)F^0_{imn}x_m x_n + \cdots] = 0. \tag{4}$$

On multiplying out, the first product vanishes and the kinematic properties of the system are determined by the remaining quadratic form. When the form is definite, all $x_n = 0$ and $X_n = X^0_n$ is an isolated solution of eqns (1); the system lacks kinematic mobility and possesses a unique configuration. For this reason, such systems are classified in Ref. [4] as "exceptional invariant systems". It seems more appropriate to call them quasi-variant, since they represent an exceptional, singular case of underconstrained systems, which in general are kinematically mobile (variant).

More recent research on underconstrained structural systems has been initially stimulated by the invention of tensegrity systems[5–8] and the proliferation of modern tensile structures[9–12]. Most of the research has been problem oriented[13, 14], with main applications to cable systems and membranes, especially their prestressed shape finding. This problem is usually solved numerically, by means of non-linear elastic analysis, although the researchers[15] are aware of the purely statical–kinematic nature of the problem. For some regular systems—axisymmetric geodesic and Chebyshev nets, and underconstrained axisymmetric 3-webs—closed form first integrals were obtained[16–18], establishing the entire set of prestressed shapes.

This paper deals with general statical–kinematic interrelations and characteristic properties of underconstrained structural systems, with an emphasis on statically indeterminate ones.

## ANALYTICAL CRITERION AND BASIC PROPERTIES OF QUASI-VARIANT SYSTEMS

The linear combination (4) of constraint equations can be modified by solving eqns (2) in terms of $(N-r)$ independent displacements $x_p$ and expressing all $N$ displacements as

$$x_n = a_{np}x_p \quad (a_{np} = 1 \text{ at } n = p). \tag{5}$$

This enables expansion (4) to be rewritten in independent displacements[11]

$$(1/2)F^0_{imn}\Lambda_i a_{mp}x_p a_{nq}x_q + \cdots = (1/2)b_{pq}x_p x_q + \cdots = 0 \tag{6}$$

with the coefficients $b_{pq}$ evaluated by summation. When the quadratic form in expansion (6) is definite or indefinite, the system is, respectively, immobile or mobile. When the quadratic form is semidefinite, higher-order terms of expansion (6) must be considered. The presence of third-order terms indicates mobility while their absence calls for the evaluation of next-order terms.

The existence of a non-trivial solution $\Lambda_i \neq 0$ is a necessary condition for any one of the constraint functions $F_i$ in eqn (1) to have an extremum compatible with the fixed values of all the remaining functions. The type of extremum depends on the character of the quadratic (or higher-order) form in expansion (6). A strict constrained minimum or maximum of just one of the $F_i$ entails the same property for all of them, and constitutes a sufficient analytical criterion for a quasi-variant system.

Physically, the existence of $\Lambda_i \neq 0$ indicates statical possibility of initial forces (self-stress). The leading term in expansion (6) is the lowest-order differential of the work done by the initial forces on virtual displacements compatible with the constraint conditions. If

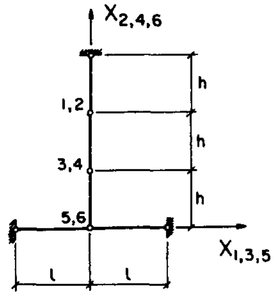


Fig. 1. Quasi-variant system (infinitesimal mechanism) of second order.

this work is strictly positive the state of self-stress is stable. Thus, an underconstrained system allowing a stable self-stress is quasi-variant.

A few clarifying remarks are in order. In what follows, the term "self-stress" is taken to mean just a non-trivial solution to homogeneous equilibrium equations, i.e. a set of statically possible initial forces. The term "state of self-stress" implies the actual state of a system. As is readily seen, it is just the statical possibility (and not the actual state) of a stable self-stress that constitutes the statical criterion of a quasi-variant system. Indeed, only the formal existence of $\Lambda_i \neq 0$ is necessary in the above criterion, and just definiteness, rather than positive definiteness, of the quadratic form (6) entails $x_p = 0$. Note that the stability in question is of kinematic, or geometric nature, since the foregoing reasoning does not involve any notion of elasticity. Thus, a quasi-variant system is only kinematically immobile and is nothing but an infinitesimal mechanism with either a finite or an infinite number of degrees of freedom. If made of a real material, such a system allows first-order displacements at the expense of second- or higher-order deformations of the structural members. In fact, the order of smallness of the constraint variations resulting in first-order displacements is a general measure of the deformability of any system, from invariant to mobile. For a quasi-variant system, this measure evolves as a by-product in analyzing the lowest-order form in expansion (6).

Finally, the generalized constraint reactions, $\Lambda_i$, appearing in the preceding equations are not the actual member forces, $N_i$. The latter are evaluated by equating two expressions for virtual work (*i*th member)

$$\Lambda_i \delta F_i = N_i \delta l_i \quad \text{(no summation)} \tag{7}$$

where $\delta l_i$ is the infinitesimal displacement corresponding to $N_i$. For a constraint preserving a certain distance, $l_0$, a convenient representation is

$$F_i(X_n) \equiv [l_i(X_n)]^2 - l_{i0}^2 = 0. \tag{8}$$

Then

$$\delta F_i = 2l_i \delta l_i \quad \text{(no summation)} \tag{9}$$

and

$$N_i = 2l_i \Lambda_i \quad \text{(no summation)}. \tag{10}$$

Note that the generalized reactions $\Lambda_i$ corresponding to constraint conditions of the form of eqn (8) are identical to the "force densities" in Ref. [12].

The criterion for a quasi-variant system based on the analysis of expression (6), is sufficient but not necessary as demonstrated by the following.

*Example.* Constraint equations for the system in Fig. 1 are

$$X_1^2+(3h-X_2)^2-h^2=0,\quad (X_5+l)^2+X_6^2-l^2=0,$$
$$(X_3-X_1)^2+(X_4-X_2)^2-h^2=0,\quad (l-X_5)^2+X_6^2-l^2=0.$$
$$(X_5-X_3)^2+(X_6-X_4)^2-h^2=0,$$

In the given configuration, $X_1^0=X_3^0=X_5^0=0$, $X_2^0=2h$, $X_4^0=h$, $X_6^0=0$, and (upon dropping the scalar factor 2) the matrix of linearized constraint eqns (2) is

$$F_{in}^0=\begin{bmatrix} 0 & -h & & & & \\ 0 & h & 0 & -h & & \\ & & 0 & h & 0 & (-h) \\ & & & & l & 0 \\ & & & & -l & 0 \end{bmatrix}. \tag{11}$$

The rank of this matrix is 4 and the solution of the system in terms of independent displacements, $x_1$ and $x_3$, is $x_2=x_4=x_6=0$, $x_5=0$. Transposing the matrix and solving the resulting system of equilibrium equations yields a non-trivial solution $\Lambda_1=\Lambda_2=\Lambda_3=0$, $\Lambda_4=\Lambda_5=\Lambda$. An attempt to construct expansion (6) in the independent displacements $x_1$ and $x_3$ leads nowhere: the quadratic form vanishes and no higher-order terms exist in the power series expansions of $F_i$.

The situation is resolved by considering a subsystem comprised of the members with non-zero constraint reactions (in this case—by the horizontal bars 4 and 5). Taking $x_6$ as an independent displacement gives rise to a one-term quadratic form, $\Lambda x_6^2$. Hence, $x_6=0$ to at least the third order and the variable $X_6$ in the constraint equations can be fixed: $X_6=X_6^0$. Then the parenthesized term in matrix (11) vanishes and the equilibrium equations obtained by transposition allow a non-trivial solution, $\Lambda_1=\Lambda_2=\Lambda_3=\Lambda^*$. The resulting quadratic form, $\Lambda^*(x_1^2+x_3^2-x_1x_3)$, is definite, so that $x_1=x_3=0$ and the entire system possesses a unique geometric configuration. However, the small displacements $x_1$ and $x_3$ would require horizontal bar elongations of only the third order of smallness. Following Koiter[19], such a system can be classified as an infinitesimal mechanism of second order.

Since the rank of the equilibrium matrix (3) for an underconstrained system is $r<N$, the system in a given geometric configuration can balance only certain external loads. These are called equilibrium loads and are representable as linear combinations of the matrix columns with arbitrary $\Lambda_i$. There is no one-to-one correspondence between the equilibrium loads and geometric configurations: there are $r$ linearly independent equilibrium loads for a given configuration, whereas, generally, there is only one equilibrium configuration for a given load. This peculiar feature of underconstrained systems underlies a unique method of obtaining precise geometric forms by statical means. Upon obtaining a desired configuration, the system can be either immobilized (by introducing additional constraints or fixing the system with a matrix) or actively controlled (by monitoring the external loads) using some kind of feedback.

An underconstrained system subjected to a non-equilibrium load must change its configuration in order to balance the load. This makes the corresponding statical problem not only geometrically nonlinear but, in a certain sense, nonlinearizable. Specifically, the load increment method widely used for non-linear problems is, as a rule, inapplicable in this situation: however small the load increment, the linearized system of equations, like the original one, allows a solution only for equilibrium loads. This is not the case with a prestressed quasi-variant system. Under a general load increment, it too must change its configuration before coming to equilibrium. Nevertheless, the presence of initial forces makes linearization feasible at least as a first step of an incremental solution. The latter exhibits two dissimilar modes of resisting external loads. In supporting an equilibrium load the stiffness is of conventional nature and the presence of prestress is irrelevant, whereas in resisting a non-equilibrium load the stiffness depends on the prestress level and may even be independent of the member properties. This question was addressed in more detail in Refs [11, 20] and, with application to cable nets, in Ref. [21].

## STATICAL DETERMINACY AND INDETERMINACY AND RELATED LOCAL AND GLOBAL PROPERTIES

Conventional definitions of a statically determinate (indeterminate) system refer to the possibility (impossibility) of determining the member forces from the equations of statics alone. These definitions and their underlying concepts have evolved in structural mechanics of geometrically invariant systems with small deformations. For geometrically non-linear systems the conventional concepts are shaky if not counterproductive, especially so for underconstrained systems. Yet, statical determinacy or indeterminacy is not just a label indicating the degree of complexity of a computer-unaided analysis. Rather, it is a very informative and even fruitful concept, reflecting the most fundamental features of structural systems (such as uniqueness or nonuniqueness of a statically possible state with a given load; behaviour under thermal action, member yielding, support settlement or imprecision in member sizes; possibility of prestress, and so on). Thus, the need for a logically consistent and universal concept of statical determinacy is obvious. It can be fulfilled by accepting the following.

*Definition.* Statical determinacy (indeterminacy) is the property of uniqueness (non-uniqueness) of a solution to the homogeneous system of equilibrium equations.

This feature, usually perceived as one of the formal manifestations of statical determinacy is, in fact, its essence and the key to the resolution of the controversy. Indeed, equations of statics pertain to the system in its final state (regardless of whether or not it is known in advance) which includes the deformed configuration of the system and the condition of each member (e.g. yielding or disengagement). Accordingly, information obtained from the equilibrium equations is characteristic of the particular state of the system and not of the system per se. As a result, some relevant parameters and properties are of local rather than global nature (in the state space), and can differ for the different states of one and the same system. By implicitly recognizing this fact the above definition (a) comprises all conceivable types of structural systems; (b) reduces to the conventional definition when applied to an invariant linear system; (c) provides a rigorous and universal measure of the degree of statical indeterminacy

$$DSI = I - r \tag{12}$$

where $I$ is the number of constraints and $r$ is the equilibrium matrix rank. In fact, $DSI$ is nothing but the number of dependent constraints, i.e. those represented by linearly dependent constraint functions. The dependence can be either global (when functions $F_i$ in eqns (1) are dependent) or local (when only the linearized functions in eqns (2) and (3) are dependent). In the latter case the matrix rank restores upon exiting from the singular configuration.

The foregoing reasoning holds for certain other attributes of the statical–kinematic analysis based on the local power series expansions (2) of constraint functions. The two most important characteristics are: the number of linearly independent equilibrium loads

$$NEL = r \tag{13}$$

and the number of degrees of freedom (the degree of kinematic indeterminacy)

$$DKI = N - r \tag{14}$$

where $N$ is the total number of generalized coordinates. Note that some or even all of the degrees of freedom comprised by $DKI$ might be realizable only as infinitesimal displacements. For a geometrically invariant system

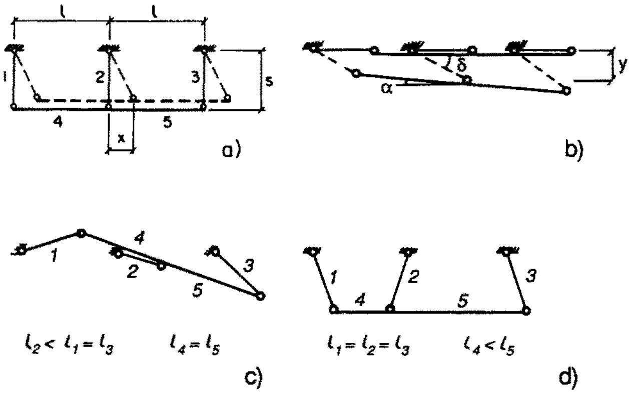


Fig. 2. Transformations of statically indeterminate finite mechanism: (a), (b) small perturbations of original and folded configurations; (c), (d) geometrically invariant, statically determinate systems resulting from bar length variations.

$$r = N \leqslant I \tag{15}$$

and the above rank-related characteristics coincide with their respective conventional counterparts.

Finally, a few curious incidents of conservation based on two global topological invariants of a structural system

$$DSI + NEL = I \tag{16}$$

$$DKI + NEL = N \tag{17}$$

$$DKI - DSI = N - I. \tag{18}$$

The assembly in Fig. 2 is a simple yet unexpectedly comprehensive example of possible situations. The constraints in this system are globally dependent and $DSI = DKI = 1$ in any ordinary configuration. The folded configuration in Fig. 2(b) is singular: the matrix rank drops, raising both $DSI$ and $DKI$ to 2. Accordingly, in this configuration the system allows two linearly independent states of self-stress (initial forces in any two bars can be assigned arbitrarily); and two independent displacements (the horizontal bar can undergo a vertical translation and an infinitesimal tilt).

Among continuous underconstrained systems, membrane shells and cable nets are the most thoroughly studied. As an example, consider a net with rhombic cells (a Chebyshev net) attached to a rigid closed contour. This is an underconstrained system with a large number of degrees of freedom. In an ordinary configuration it is statically determinate, with just one statically possible state for each equilibrium load. However, the net allows numerous singular configurations in which it becomes statically indeterminate and, possibly, quasi-variant.

A most interesting and rich class of singular configurations of a Chebyshev net is the class of translation nets (Fig. 3). This type of net was shown to be statically indeterminate to the first degree and the self-stress was obtained in a closed form[11]. When the surface

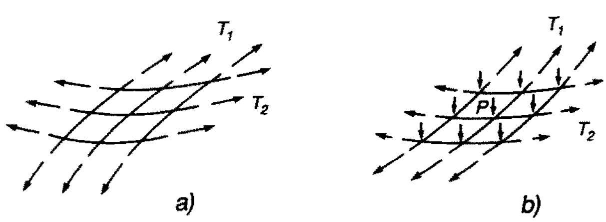


Fig. 3. Statically indeterminate Chebyshev net: (a) state of self-stress in an anticlastic net; (b) one of the statically possible states for a given load in a sinclastic net.

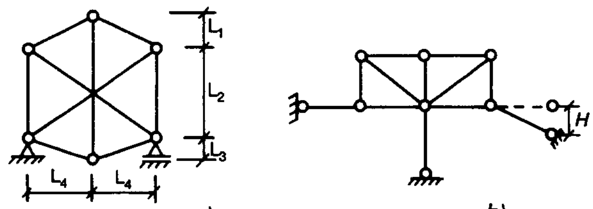


Fig. 4. Ordinary and singular configurations: (a) infinitesimal mechanism with internal indeterminacy ($L_1 = L_3$); (b) infinitesimal mechanism with external indeterminacy ($H = 0$).

is anticlastic (saddle-shaped) (Fig. 3(a)), the net can be prestressed. For a sinclastic (concave) net (Fig. 3(b)) the self-stress is unrealizable but in other ways the indeterminacy is quite tangible. It can be utilized by choosing and implementing the force ratio, $T_1/T_2$, in the two cable arrays which is most favorable for the support structure. Asymptotic nets of anticlastic surfaces are quasi-variant and indeterminate to a high degree. A prominent example of this class is a segment of an axisymmetric Chebyshev net (like a basketball net) stretched between two edge rings, which has the form of a pseudosphere[17]. Finally, a flat Chebyshev net is, obviously, quasi-variant.

## KINEMATIC MOBILITY AND SELF-STRESS

This topic is closely related to the following question raised by Tarnai[22]: What criterion determines whether self-stress stiffens an assembly which is both statically and kinematically indeterminate?

It appears that this criterion is the stability of self-stress. Moreover, it can be stated that any physically realized self-stress in a kinematically indeterminate system has a stiffening effect. Indeed, to be realizable, a statically possible, self-equilibrated state must be at least locally stable, and then, by definition, the system will resist any sufficiently small perturbation. Applying this observation to an assembly with only local statical and kinematic indeterminacy is straightforward. Such a system is adequately constrained and, upon exiting from the singular configuration (due to some variation in geometry), becomes determinate and geometrically invariant. Examples in Fig. 4 represent the two basic types of such systems—one with internal (at $L_1 = L_3$) and the other with external (at $H = 0$) indeterminacy. In their singular configurations both systems are infinitesimal mechanisms and allow a stable state of self-stress. These configurations can be analytically detected by Timoshenko's unit load test which is, in effect, a check of the equilibrium matrix rank and conceptually can be traced back to Mobius[1].

Going back to underconstrained systems, recall that these are globally kinematically indeterminate but allow singular configurations where they acquire statical indeterminacy as well. If the possible self-stress is unstable the system is a finite mechanism in a singular ("dead center") configuration; the possibility of a stable self-stress characterizes a quasi-variant system (a multifreedom infinitesimal mechanism). The work of initial forces (6) is also the work of external perturbation forces on the same virtual displacements. The composition of the leading term in expansion (6) shows that the system resistance to perturbations, obtainable by differentiation, is always proportional to the magnitude of initial forces, $\Lambda_i$, but can be nonlinear in displacements. Accordingly, a stable state of self-stress has a stiffening effect, which is of the same order as the order of the infinitesimal mechanism. In particular, if the mechanism is higher than first order, the tangent stiffness matrix is singular, yet the stiffening effect exists.

A curious and ostensibly paradoxical type of structural system is an assembly with global statical and kinematic indeterminacy, i.e. a finite mechanism apparently allowing self-stress in any configuration. For the planar assembly in Fig. 2(a), the initial forces are easily found but expression (6) vanishes. Thus, statical–kinematic analysis treating structural members as undeformable shows only that the system is kinematically mobile and that self-stress is statically possible. In investigating this system, two distinct situations must be addressed. First, prestressing may change the natural (strain-free) member sizes such that the assembly ceases to be a finite mechanism. For example, if the horizontal bar bends, the system loses kinematic mobility and becomes an infinitesimal mechanism. A more interesting situation arises when the assembly retains kinematic mobility in its prestressed configuration, i.e. has the appropriate member sizes after prestress. This is the case if the horizontal bar either originally has a camber or is straight and perfectly rigid in bending (but not in compression). Assuming the second alternative, hereafter the bar is modelled as a spring sliding along a rigid rod.

Examining this kind of prestressed finite mechanism, note, first of all, that stable equilibrium must be ruled out. Indeed, since purely kinematic motion preserves the member sizes and forces, it requires no increase in the elastic strain energy, which in the absence of external loads could be the only source of a restoring force. Hence, a prestressed finite mechanism is at best at neutral equilibrium. However, even this is feasible only in exceptional trivial cases of rigid motion (e.g. a tensioned chain with one end hinged and the other sliding along a circular guide). In general, a kinematic perturbation upsets equilibrium of the existing member forces, since their nodal resultants will no longer add up to zero. This means that adjacent equilibrium states are absent and the self-stress, although statically possible, is unstable.

Thus, finite kinematic mobility and state of self-stress are incompatible: they cannot exist in one and the same assembly simultaneously. Accordingly, any prestressed assembly lacks finite kinematic mobility and is, at most, an infinitesimal mechanism; conversely, a system with finite kinematic mobility cannot be prestressed and retain its mobility. In the light of this conclusion it is obvious, for example, that self-stress is unrealizable for a class of statically indeterminate finite mechanisms with cyclic and reflection symmetries studied in Ref. [23].

The foregoing observations can be summed up in the following statement, which constitutes the answer to Tarnai's question. A statically and kinematically indeterminate system with a given fixed geometry can be in a state of self-stress if, and only if, it is an infinitesimal mechanism; in this case the state of self-stress has a stiffening effect of the same order as the order of the mechanism.

To illustrate the entire variety of possible situations and to quantify the preceding statements, consider again the assembly in Fig. 2(a). Let the statically possible self-stress

$$N_1 = N_3 = N_0, \quad N_2 = -2N_0, \quad N_4 = N_5 = 0$$

be induced while restraining the system. Upon its release, the system will fold up. Equilibrium in the folded configuration requires the elastic nodal displacements in the horizontal direction

$$x_1 = x_3 = -x_2/2 = N_0/(3k_l+k_s) = x \tag{19}$$

so that the member forces and lengths become

$$\begin{gathered} N_1 = N_3 = -N_2/2 = N_4 = -N_5 = 3k_l N_0/(3k_l+k_s) = N \\ l_1 = l_3 = s(1-\xi), \quad l_2 = s(1+2\xi), \quad \xi = x/s \\ l_4 = l(1+3\eta), \quad l_5 = l(1-3\eta), \quad \eta = x/l. \end{gathered} \tag{20}$$

Here $k_l = l/EA_l$ and $k_s = s/EA_s$ are the stiffness parameters for the horizontal bar and support bars, respectively. Note that $\xi$ and $\eta$, however small they may be, are finite.

To investigate the stability of the state (20) the system is given a small perturbation and the elastic strain energy increment is evaluated

$$\Delta U = N_i(\varepsilon l)_i + \tfrac{1}{2}k_i(\varepsilon l)_i^2.$$

Within the required accuracy the elastic strains are

$$\varepsilon_i = (x_i + y_i^2/2l_i)/l_i \quad (i = 1, 2, 3;\ \text{no summation})$$
$$\varepsilon_4 = \alpha^2/2 + (x_2 - x_1)/l_4, \quad \varepsilon_5 = \alpha^2/2 + (x_3 - x_2)/l_5.$$

Since the horizontal bar does not bend (Fig. 2(b))

$$y_1 = y - \alpha l, \quad y_3 = y + \alpha l$$

and only five out of the six nodal displacements are independent. After the necessary substitutions and rearrangements with consistently maintained accuracy, and upon the introduction of $\delta = y/s$, $\Delta U$ becomes

$$\Delta U = 3Ns\xi(\delta - \alpha)^2 + N(\alpha l)^2/s + k_s(x_1^2 + x_2^2 + x_3^2)/2 + k_l[(x_2 - x_1)^2 + (x_3 - x_2)^2]/2. \quad (21)$$

At $N > 0$ or, which is the same, $N_0 > 0$, form (21) is positive definite. Thus, with the middle bar in compression, the folded configuration is stable and the system is a self-stressed infinitesimal mechanism; purely kinematic displacements are impossible, but first-order elastic displacements require only second-order strains.

At $N < 0$, form (21) is indefinite and the folded configuration of Fig. 2(b) is unstable. In this case the combination of natural bar lengths

$$l_2^0 < l_1^0 = l_3^0, \quad l_4^0 = l_5^0$$

is compatible, and the system acquires a configuration (Fig. 2(c)) which is geometrically invariant and statically determinate (hence, stress free). Although the system can be forced into the self-stressed configurations shown in Figs 2(a) and (b), it will not stay in either of them. Similarly, a set of bars with

$$l_1^0 = l_2^0 = l_3^0, \quad l_4^0 < l_5^0$$

can be assembled (Fig. 2(d)) to form an invariant system. It can be forced into two distinct folded self-stressed configurations—left and right, both of them unstable. Thus, different combinations of the bar lengths can render the considered system invariant, variant or quasi-variant.

Taking advantage of eqn (21) the response of the prestressed assembly (Fig. 2(b)) to a vertical load, $V$, can be evaluated

$$\partial U/\partial \alpha = 2Nl^2\alpha/s - 6Ns\xi(\delta - \alpha) = 0$$
$$\partial U/\partial \delta = 6Ns\xi(\delta - \alpha) = V$$

from which

$$\alpha = Vs^2/2Nl^2, \quad \delta = V/6N\xi.$$

Expressing $N$ and $\xi$ in terms of the original prestressing force with the aid of eqns (19) and (20) gives

$$V = \frac{18N_0^2\delta}{sk_l(3 + k_s/k_l)^2}. \quad (22)$$

The structural response exemplified by solution (22) is weak, but otherwise quite remarkable: the resistance of the system is proportional to the square of the prestressing force $N_0$ and reduces with the increasing stiffness of the horizontal bar. This contrasts with

Table 1

| System description and load type | Resistance to $\delta$ due to Prestress | Member stiffness |
|---|---|---|
| Invariant system, arbitrary load; any system, equilibrium load | $N_0\delta$ | $EA\delta$ |
| Quasi-variant system (first order), non-equilibrium load | $N_0\delta$ | $EA\delta^3$ |
| "Prestressed finite mechanism", non-equilibrium load | $N_0^2\delta/EA \approx \varepsilon_0 N_0\delta$ | |

the response of other systems with a similar prestress level and typical member stiffness, $EA$. Table 1 compares three types of systems in terms of respective external loads producing the same non-dimensional displacement $\delta$.

In the third case, both the infinitesimal mechanism and the stiffening effect of self-stress are first order. This is seen from the right-hand side of the last expression (assuming the elastic strain $\varepsilon_0$ is finite).

## CONCLUSION

A peculiar combination of interrelated statical and kinematic properties governs the behavior of underconstrained systems, in particular, statically indeterminate ones. The most prominent potential feature is the statical possibility of a stable self-stress which is shown to be a necessary and sufficient criterion of a quasi-variant system. However, even when the state of self-stress is unstable and, therefore, physically unrealizable, statical indeterminacy in underconstrained systems is a useful property. For a system of a given structural topology, different combinations of member lengths can render the system invariant, variant or quasi-variant. Furthermore, different geometric configurations of a variant system may differ in their respective degrees of statical and kinematic indeterminacy and the number of independent equilibrium loads. The intricate interrelation between the equilibrium loads and configurations underlies the concept of statically controlled geometry which is instrumental in the obtainment and active control of precise geometric shapes.

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